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A Study of Minkowski Space with Time Topology

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Abstract—*In 1967, Zeeman proposed time topology on Minkowski space, the spacetime of Special Theory of Relativity. Time topology is defined as the finest topology on Minkowski space that induces Euclidean topology on every timelike straight line. In the present paper, compactness and connectedness of certain sets in Minkowski space have been studied in the context of time topology.*

Keywords: *Compactness, Connectedness, Minkowski space, Time topology*

1. INTRODUCTION

Minkowski space, an arena of Einstein's special theory of relativity, represents a mathematical model of the spacetime. There are several topologies defined on it. The well known Euclidean topology on the Minkowski space is a natural topology but it was supposed to be an inappropriate choice because it does not incorporate the causal structure of spacetime and its homeomorphism group is too large to be of any physical significance. In 1967, Zeeman [10] introduced non-Euclidean topologies on Minkowski space which include fine topology, *t*-topology, *s*-topology, space topology, time topology etc. and studied fine topology. Zeeman obtained that the homeomorphism group of Minkowski space with the fine topology is isomorphic to the group generated by the Lorentz group, translations and dilatations. Nanda [8, 9] studied the homeomorphism group of Minkowski space with *t*, *s* and space topologies. In 1976, Göbel [3] generalized the fine topology on manifolds and studied its homeomorphism group and Hawking et.al. [5] proposed and studied the path topology on strongly causal spacetime. Further, Göbel [4] explored a physically relevant topology on spacetime and proved Zeeman's [10] conjecture for the homeomorphism group of Minkowski space with time topology. In 2009, Agrawal and Shrivastava [1] studied compact sets of Minkowski space with the *t*-topology which is same as the path topology on Minkowski space. Low [6], in 2010, proved the non-simple connectedness of spacetime manifold with the path topology. Further, in the same year Agrawal and Shrivastava [2] provided an alternative proof for the non-simple connectedness of Minkowski space with the path topology. For the Euclidean space, the compact and connected subspaces are well studied. The present paper is focussed on the study of topological notions, namely compactness and connectedness, for some sets in Minkowski space with the time topology.

2. NOTATION AND PRELIMINARIES

Let the set of natural and real numbers be denoted by N and R respectively. For $n \in N$ and $n > 1$, the n-dimensional real vector space R^n with the bilinear form $g: R^n \times R^n \to R$ such that g is symmetric, nondegenerate and there exists a basis $\{e_0, e_1 \dots e_{n-1}\}$ for R^n with $g(e_i, e_j) = 1$ if $i = j = 0$, -1 if $i = j = 1, \dots, n-1$ and 0 otherwise is called the n-dimensional Minkowski space, denoted by M^n and the bilinear form g is called the Lorentz inner product. Also g induces an indefinite characteristic quadratic form Q on M^n defined as $Q(x) = g(x,x)$, $x \in M^n$. According as $Q(x)$ is positive, zero, or negative, $x \in M^n$ is called timelike, lightlike or spacelike. The sets $C^T(x) = \{y \in M^n : y = x \text{ or } Q(y-x) > 0\}$, $C^L(x) = \{y \in M^n : y = x \text{ or } Q(y-x) = 0\}$ and $C^S(x) = \{y \in M^n : y = x \text{ or } Q(y-x) < 0\}$ are respectively called the *time cone*, *light cone* and *space cone* at *x*. Let the coordinates of $x \in M^n$ with respect to the basis $\{e_0, e_1 \dots e_{n-1}\}$ be denoted by x^i, where $i = 0, 1, \dots, n-1$, x^0 is the time coordinate and $x^1, x^2, \dots x^{n-1}$ are the space coordinates. Then the sets $C^{T+}(x) = \{y \in C^T(x) : y^0 > x^0\}$ and $C^{T-}(x) = \{y \in C^T(x) : y^0 < x^0\}$ are called the future and past timecones at $x \in M^n$. The elements of $C^{T+}(x)$ and $C^{T-}(x)$ are called future and past directed timelike vectors respectively. A straight line parallel to a timelike vector is called a *timelike line*. Similarly, the notions of *lightlike line* and *spacelike line* in M^n are defined [7].

Let $x \in M^n$ and $B \equiv \{N_\varepsilon^E(x) : \varepsilon > 0\}$. Then the topology generated by the basis B is called the *Euclidean topology* on M^n and the topology generated by the local base $N_\varepsilon^t(x) = N_\varepsilon^E(x) \cap C^T(x)$ is called the *t*-topology on M^n. The *t*-topology is finer than the Euclidean topology [1].

Further, the *time topology* on M^n is the finest topology that induces Euclidean topology on every timelike line [10]. The time topology is finer than the *t*-topology [9]. Let M_E^n, M_t^n and M_T^n denote the n-dimensional Minkowski space with Euclidean topology, *t*-topology and time topology respectively.

3. NON-COMPACT SETS

In the present section, compactness of some sets in M^4 with time topology has been studied.

Proposition 3.1: Let M^4 be the 4-dimensional Minkowski space. Then the following sets are not compact in M_T^4: (i) a straight line L, (ii) M^4 (iii) M^4- {p}, where $p \in M^4$ and (iv) M^4 - L.

Proof: (i) Since L is not compact in M_E^4 and the time topology is finer than the Euclidean topology, L is not compact in M_T^4.

(ii), (iii) and (iv): Proof is similar to that of (i).

Proposition 3.2: Let M^4 be the 4-dimensional Minkowski space and S^3 be the unit 3-sphere in M^4. Then S^3 is not compact in M_T^4.

Proof: Let $p \equiv (0,0,0,0)$ and $z \equiv (1,0,0,0)$. Then for $k \in N$ and for some $\varepsilon > 0$, choose a sequence $z_k \in C^{T+}(p) \cap S^3$ such that $0 < d(z, z_k) < \varepsilon$ and $z_i \neq z_j$, where $d(z, z_k)$ denotes the Euclidean distance between z and $z_k, i, j \geq 1$ and $i \neq j$. Then $\{z_k\}_{k \in N}$ is a Zeno sequence in M_t^4. It is known that a set is compact in M_t^4 iff it is compact in M_E^4 and it does not contain the image of a Zeno sequence in M_t^4 [1]. Therefore S^3 is not compact in M_t^4. The time topology being finer than *t*-topology [9], S^3 is not compact in M_T^4.

4. CONNECTED AND DISCONNECTED SETS

In the present section, connectedness of some sets in M^4 with time topology has been studied.

Proposition 4.1: Let M^2 be the 2-dimensional Minkowski space with the time topology and L be a straight line in M^2. Then M^2 - L is not connected in M_T^2.

Proof: Since M_E^2- L is not connected and the time topology is finer than the Euclidean topology, M^2 - L is not connected in M_T^2.

Proposition 4.2: Let M_t^4 be the 4-dimensional Minkowski space with the time topology. Then a timelike line is connected in M_t^4.

Proof: It follows from the fact that the time topology induces Euclidean topology on every timelike straight line.

Proposition 4.3: Let M_t^4 be the 4-dimensional Minkowski space with the time topology. Then spacelike and lightlike lines are not connected in M_t^4.

Proof: Since the *t*-topology induces discrete topology on every spacelike line and on every lightlike line [1] and time topology is finer than the *t*-topology [9], spacelike and lightlike lines are not connected in M_t^4.

Proposition 4.4: Let M^4 be the 4-dimensional Minkowski space and $p \in M^4$. Then M^4 -{p} is connected in M_t^4.

Proof: Let $\gamma_{ab}: I \equiv [0,1] \rightarrow M^4 - \{p\}$ be defined as $\gamma_{ab}(t) = (1-t)a + tb$, where $t \in I$ and $a, b \in M^4 - \{p\}$. Further, let $, z \in M^4 - \{p\}$. Then $z - y$ is a timelike, lightlike or a spacelike vector. Case 1. $z - y$ is a timelike vector. Then either $p \in \gamma_{yz}(I)$ or $p \notin \gamma_{yz}(I)$. If $p \notin \gamma_{yz}(I)$, $[\gamma_{yz}(I)]^E = [\gamma_{yz}(I)]^T$ and hence γ_{yz} is a path joining y to z else choose $q \in C^T(y) \cap C^T(z) \cap (M^4 - T)$, where T is a timelike line containing p, y and z. Then $q - y$ and $q - z$ are timelike vectors and hence the join of γ_{yq} and γ_{qz} is a path from y to z in $M^4 - \{p\}$. Case 2. z$-y$ is a lightlike or a spacelike vector. Choose $w \in C^T(y) \cap C^T(z) \cap (M^4 - \{p\}$. Then w-y and $w - z$ are timelike vectors and hence the join of γ_{yw} and γ_{wz}, is a path from y to z in $M^4 - \{p\}$. This proves that $M^4 - \{p\}$ is path connected in M_t^4. This completes the proof.

Proposition 4.5: Let M^4 be the 4-dimensional Minkowski space and L be a straight line in M^4. Then $M^4 - L$ is connected in M_t^4.

Proof: Let $\gamma_{ab}: I \equiv [0,1] \rightarrow M^4 - L$ be defined as $\gamma_{ab}(t) = (1-t)a + tb$, where $t \in I$ and $a, b \in M^4 - L$. Further, let $y, z \in M^4 - L$. Then either $\gamma_{yz}(I) \cap L = \emptyset$ or $\gamma_{yz}(I) \cap L \neq \emptyset$. Case 1. $\gamma_{yz}(I) \cap L = \emptyset$. Then $z - y$ is a timelike, lightlike or a spacelike vector. If $z - y$ is a timelike vector, then $[\gamma_{yz}(I)]^E = [\gamma_{yz}(I)]^T$ and hence γ_{yz} is a path joining y to z in $M^4 - L$ else choose $q \in C^T(y) \cap C^T(z)$. Then join of γ_{yq} and γ_{qz} is a path from y to z in $M^4 - L$. Case 2. $\gamma_{yz}(I) \cap L \neq \emptyset$. Choose $q \in C^T(y) \cap C^T(z) \cap (M^4 - P)$, where P is the plane containing L, x and y. Then $q - y$ and $z - q$ are timelike vectors and hence join of γ_{yq} and γ_{qz} is a path from y to z in $M^4 - L$. This proves that $M^4 - L$ is path connected and hence connected in M_t^4.

Proposition 4.6: Let M^4 be the 4-dimensional Minkowski space and S^3 be the unit 3-sphere in M^4. Then S^3 contains more than one isolated points and hence S^3 is not connected in M_t^4.

Proof: Let $P \equiv S^3 \cap C^T(0,0,0,0)$. Then P contains more than one points. Now for all $x \in P$ and for every $\varepsilon > 0$, $N_\varepsilon^t(x) \cap P = \{x\}$. This proves that $P \subset S^3$ contains more than one isolated points. Hence the result.

Proposition 4.7: Let M^4 be the 4-dimensional Minkowski space and S^3 be the unit 3-sphere in M^4. Then for $x \in S^3$, $S^3 - \{x\}$ is not homeomorphic to 3-dimensional Euclidean space.

Proof: As proved in Proposition 4.6, $S^3 - \{x\}$ contains isolated points while the 3-dimensional Euclidean space is connected. Hence the result.

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